NASA TECHNICAL TRANSLATION NASA TT F-15947

ACCURATE MATHEMATICAL MODELLING OF PWM POWER REGULATORS

R. Prajoux, J.C. Marpinard, and J. Jalade

Translation of Report, Laboratoire d'Automatique et d'Analyse des systemes, C.N.R.S., Issued under Estec/Contract No. 2026 AK, May 1974, 25 pp.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 SEPTEMBER 1974

STANDARD TITLE PAGE

1. Report No. TT F-15947	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle ACCURATEGMATHEMATICAL MODEELING OF PWM POWER REGULATORS: PARTIAL REPORT		5. Report Date September 1974
		6. Performing Organization Code
7. Author(s) R. Prajoux, J.C. Marpinard, J. Jalade Automatics and Systems Analysis Laboratory, National Scientific Research Center, Toulouse		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, CA 94063		11. Contract or Grant No. NASw-2481
		13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes
Translation of Report, Laboratoire d'Automatique et d'Analyse des Systemes, C.N.R.S., Issued under Estec/Contract No. 2026 AK, May 1974, 25 pp.

16. Abstract
A method is described which makes it possible to put the dynamic behavior of PWM converters into equation form and to determine their pulsed transmittance for any given structure or mode of operation. Simplifications for practical application of the method are performed a posteriori to clarify their range.

17. Key Words (Selected by Author(s))	18. Distribution Statement
	Unclassified-Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassifiedc	31	

ACCURATE MATHEMATICAL MODELLING OF PWM POWER REGULATORS

R. Prajoux, J.C. Marpinard and J. Jalade
Automatics and Systems Analysis Laboratory
National Scientific Research Center, Toulouse

General Remarks

The increasingly widespread use of regulator loop PWM converters has led to consideration of the problem of the dynamic stability of the looped system. /*

Since this type of system is a discrete non-linear system, various approaches have been taken in attempting to analyze its operation.

The principal work in this area has made use of:

-- continuous linear approximation [1];

-- the first harmonic method [2];

-- the phase plane method with possible linearization by bits [3].

These methods have the advantages of being relatively simple and of using the results of continuous linear servo theory, but, by failing to take the discreteness of the system into account, they do not allow for fine analysis of its behavior. Intorder to move in closer to reality, therefore, the method of the recurrence related to the system may be usedd [4]. This method offers the advantage of describing the strong signal behavior; on the other

*Slash marks in the margin indicate a new page in the foreign text.

hand, it is poorly suited for synthesis and quickly becomes relatively cumbersome, especially if the order of the system increases.

Small signal analysis, using a z transform, for example, makes it possible, taking the discrete nature of the system into account, to describe not only the local stability but also the transient response, while at the same time placing the main parameters of the system in condensed letter form.

Let G(p) (Fig. 1) represent the transmittance of a given continuous linear system T. If this system is subjected to a sample input $e^*(t)$, its z transmittance, $\overline{G}(z)$, is defined in such a way that:

$$\overline{S}(z) = \overline{G}(z)\ \overline{E}(z)$$

One of the forms of $\overline{G}(z)$ is known to be:

$$\left|\begin{array}{l} \overline{G}(z) = \sum_{n=0}^{\infty} g(n\,T)\ .\ z^{-n} \\ g(t) = \text{pulse response} = \mathcal{L}^{-1}\left[G(p)\right] \end{array}\right. \qquad (1)$$

Now let T_1 represent a continuous or non-continuous linear system. If T_1 is such that its discrete pulse response $g_1(n\ T)$ is:

$$g_1(n\ T) = g(n\ T),$$

its z transmittance $\overline{G}_1(z)$ is:

$$\overline{G}_1(z) = \overline{G}(z) \qquad (2)$$

As a result, if T_1 is subjected to the discrete input $\overline{E}(z)$, the same discrete output S(z) is observed as for T.

The discrete behavior of any linear system of discrete pulse response g(n T) is equivalent to that of a continuous linear system of pulse response g(t).

If the behavior of the continuous system T is described by the state equations:

$$\left| \begin{array}{l} \overset{\circ}{\underline{Z}} = A\,\underline{Z} + \underline{B}\,u(t) \\ s(t) = C\,\underline{Z} \end{array} \right. \tag{3}$$

one has

$$\left| \begin{array}{l} g(nT) = C.\ e^{n\,AT}\ .\ B = C\ .\ \phi^{n}(t)\ .\ B \\ \phi(T) = \text{matrix of discrete transition state} \end{array} \right. \tag{4}$$

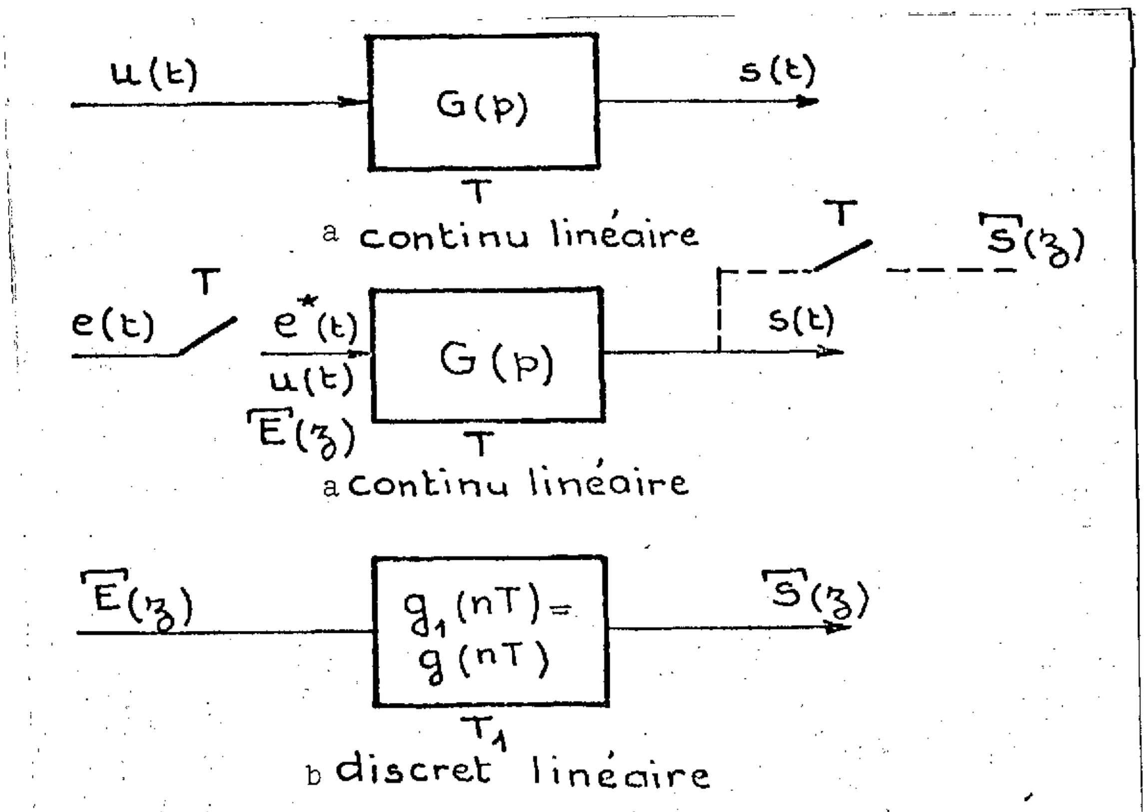


Fig. 1.
Key: a. Linear continuous, b. Linear discrete.

One may obtain G(z) for any linear discrete system by the procedure;

$$\left.\begin{matrix} C \\ \phi\,(T) \\ B \end{matrix}\right] \longrightarrow g\,(n\,T) \longrightarrow g\,(t) \longrightarrow \overline{G}\,(z)$$

The change from G(t) to $\overline{G}\,(z)$ may conveniently be performed with the use of ordinary transform tables.

We will now discuss determination of the equivalent continuous system of a PWM converter by the discrete pulse response method.

Behavior of a Converter

Under certain operating conditions, the structure of the electrical network representing the system to be studied may be changed (blocking or saturation of transistors, current becoming cancelled in one branch of the network for a fraction of the p period, etc.). It follows that the continuous equations governing the behavior of the different magnitudes must change over time, and that the converter may not be characterized by a continuous transmittance which is valid at any time. However, <u>the converter is still a discrete system</u>, since:

-- it is controlled by a series of discrete instants t_n;

-- the output of the system (assuming that the loop has been opened) is also a series of discrete instants t'_n (Fig. 2).

In addition, this will be a <u>linear</u> discrete system if its weak signal behavior is being studied; the previously described method may thus be applied to it.

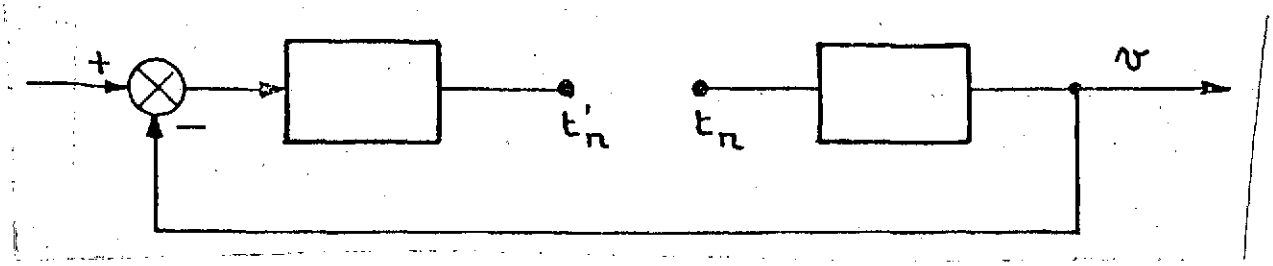
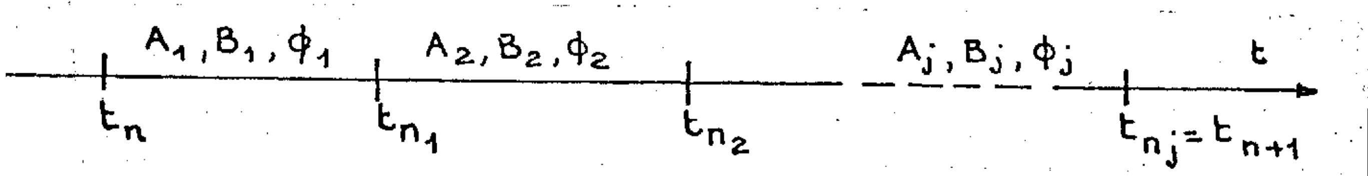


Fig. 2.

Discrete Pulse Response of Systems with Changes in Structure

The period of operation is assumed to be divided into j time intervals, each of these corresponding to a different structure of the system.

During each of these intervals the behavior of the system is that of a linear continuous system:



The switching instants t_n, which by definition are <u>control instants</u>, are determined in closed loop configuration by an equation of the type:

$$g_n(\underline{Z}_n) = a(t_n) \quad \text{with} \quad \underline{Z}_n = \underline{Z}(t_n).$$

when a coincidence modulator is involved.

With an open loop, the instants defined in this manner are the t'_n, the t_n instants being by definition known a priori.

The switching instants t_{ni} within the period $[t_n, t_{n+1}]$ are defined by equations of the type:

$$h_i(\underline{Z}_{ni}) = 0 \quad \text{with} \quad \underline{Z}_{ni} = \underline{Z}(t_{ni}).$$

These will be termed structure change instants.

For the j successive states, the state equations of the system (cf. that of Fig. 3, for example) are in the form:

$$\left|\begin{array}{lll} \dot{\underline{Z}} = \underline{A}_1\ \underline{Z} + \underline{B}_1\ u & \text{for} & t_n < t < t_{n_1} \\ \dot{\underline{Z}} = A_2\ \underline{Z} + B_2\ u & \text{for} & t_{n_1} < t < t_{n_2} \\ \text{-------------} & & \\ \dot{\underline{Z}} = A_j\ \underline{Z} + B_j\ u & \text{for} & t_{n(j-1)} < t < t_{n+1} \\ \text{and} & & \\ V = C\ \underline{Z} & \text{for any value of } t & \end{array}\right|$$

The general solution for a given structure k is in the form:

$$\left|\begin{array}{l} \underline{Z}(t) = \phi_k(t-t_0).\underline{Z}(t_0) + \int_{t_0}^{t} \phi_k(t-\lambda).B_k.\underline{u}(\lambda).\ d\lambda \\ \text{with} \\ \phi_k(t-t_0) = e^{A(t-t_0)} \end{array}\right| \qquad (6)$$

that is:

$$\underline{Z}(t) = \phi_k(t)\left[\phi_k(-t_0).Z(t_0) + \int_{t_0}^{t} \phi_k(-\lambda).B_k.\underline{u}(\lambda).\ d\lambda\right] \qquad (7)$$

If the state vector $\underline{Z}(t)$ is considered only at switching instants, that is, $\underline{Z}(t_n)$ and $\underline{Z}(t_{ni})$, this state vector necessarily does not constitute a discrete state vector. For example, the result of having a $\underline{Z}_{ni}$ such that

$$h(\underline{Z}_{ni}) = 0$$

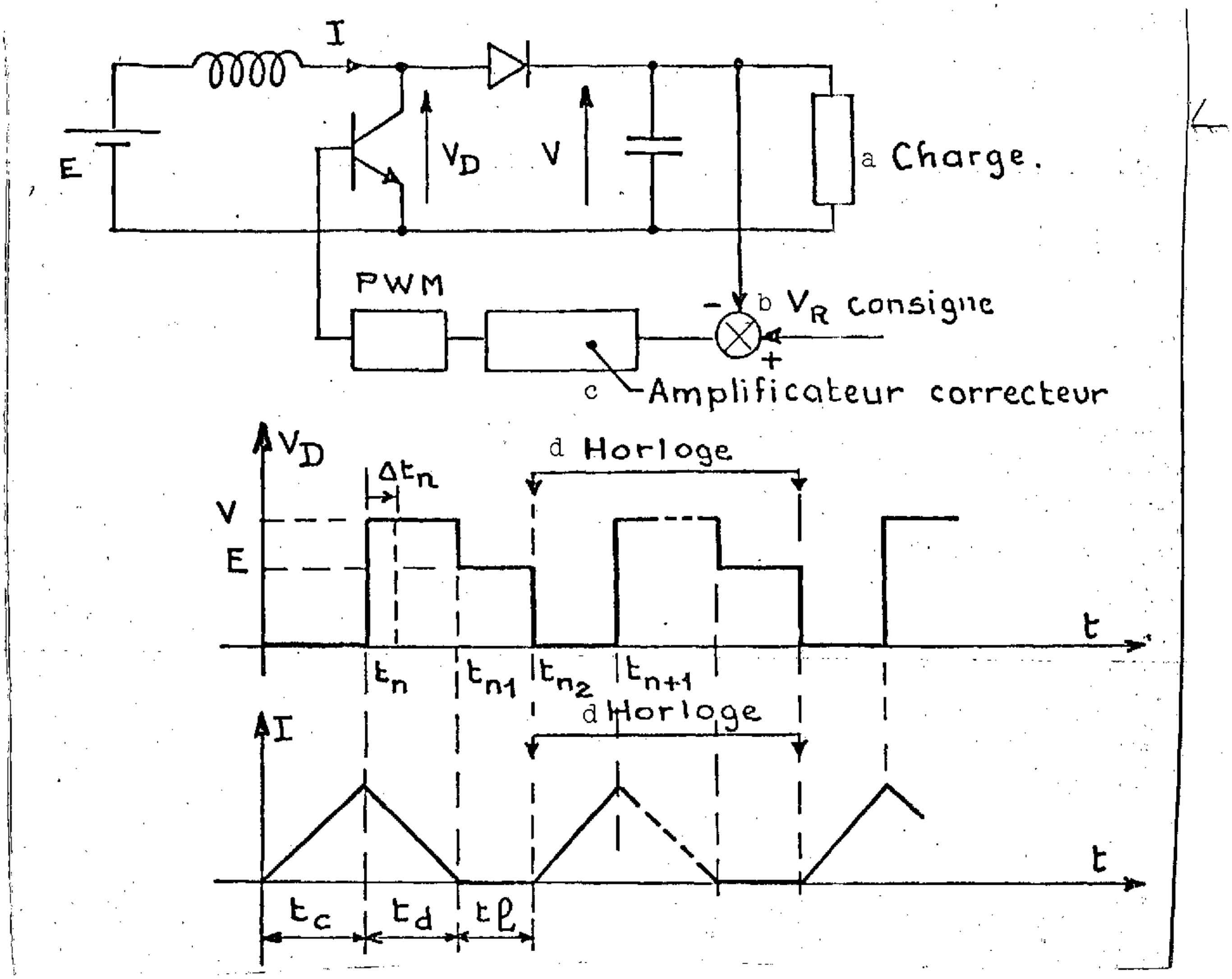


Fig. 3.

Key: a. Load.
b. V_R mandatory instruction.
c. Correcter amplifier.
d. Clock.

is that, knowing only $\underline{Z}_{ni}$, it is not possible to determine t_{ni}.

In order to have a discrete state vector, in general one will consider the state vector $\underline{Z}(t)$ at switching instants t_n or t_{ni} in continuous operation, even in disturbed operation. Assuming:

$$\underline{Z}(t) = f_{i+1}(t) \quad \text{for} \quad t_{ni} < t < t_{n(i+1)} ,$$

with $\underline{f}_{i+1}(t)$ defined for $t \geq t_{ni}$, the discrete state vector of the disturbed system may be properly represented by;

$$\underline{Z}^{*}_{ni} = \underline{f}_i(t^0_{ni}).$$

Thus for the component Z_ℓ of the state vector $\underline{Z}$, we have the notation given in Fig. 4.

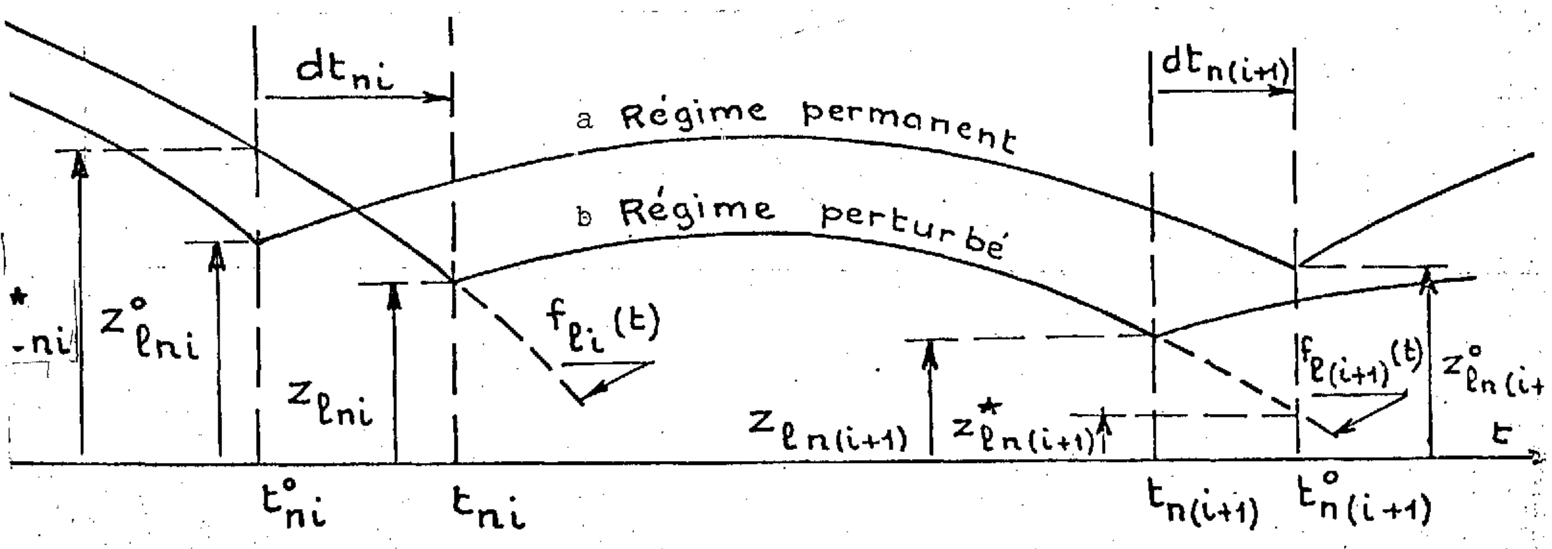


Fig. 4.

Key: a. Continuous operation.
b. Disturbed operation.

With this notation:

$$\underline{Z}^{*}_{n(i+1)} = \phi_{(i+1)}\left(t^{o}_{n(i+1)}\right)\left[\phi_{(i+1)}(-t_{ni})\,\underline{Z}_{ni} + \int_{t_{ni}}^{t^{o}_{n(i+1)}} \phi_{(i+1)}(-\lambda)\,.\,B_{(i+1)}\,.\,u(\lambda)\,.\,d\lambda\right] \qquad (8)$$

$t_{n\,(i+1)}^{0}$ being the switching instant of order i + 1 in the interval $[t_n, t_{(n+1)}]$ calculated for undisturbed operation.

In this expression, t_{ni} and $\underline{Z}_{ni}$ (a switching instant of order i in disturbed operation and the state vector at this instant in disturbed operation, respectively) are functions of $\underline{Z}^{*}_{ni}$. One may therefore write:

$$\frac{d\underline{Z}^{*}_{n(i+1)}}{d\underline{Z}^{*}_{ni}} = \phi_{i+1}\,(t^{0}_{n(i+1)}) \left[\frac{d\phi_{(i+1)}(-t_{ni})}{dt_{ni}}\,\underline{Z}_{ni}\,\frac{dt_{ni}}{d\underline{Z}^{*}_{ni}} + \phi_{i+1}\,(-t_{ni})\,\frac{d\underline{Z}_{ni}}{d\underline{Z}^{*}_{ni}} - \phi_{(i+1)}(-t_{ni}).\,B_{(i+1)}.\,u(t_{ni})\,\frac{dt_{ni}}{d\underline{Z}^{*}_{ni}} \right] \tag{9}$$

<u>This general equation holds true no matter what the amplitude of the disturbance may be, provided that the same qualitative behavior persists</u> (number of structure remaining unchanged). In the hypothesis of weak signal behavior over a span of continuous operation, this expression becomes, all computations performed (cf. App. I):

$$\left(\frac{d\underline{Z}^{*}_{n\,(i+1)}}{d\underline{Z}^{*}_{ni}}\right)^{0} = \phi^{0}_{(i+1)} \left[\left[\left(\frac{d\underline{Z}}{dt}\right)_{t^{0-}_{ni}} - \left(\frac{d\underline{Z}}{dt}\right)_{t^{0+}_{ni}} \right] \left(\frac{dt_{ni}}{d\underline{Z}^{*}_{ni}}\right)_{t^{0}_{ni}} + \mathbb{1} \right] \tag{10}$$

with

$$\phi_{(i+1)}^{0} = \phi_{(i+1)}\,(t^{0}_{n\,(i+1)} - t^{0}_{ni})$$

This equation is of the form:

$$\frac{d\underline{Z}^{0}_{n\,(i+1)}}{d\underline{Z}^{*}_{ni}} = \phi^{0}_{(i+1)} \left[K_i + \mathbb{1} \right] \tag{11}$$

One may now express dZ^*_{n+1}/dt_n, which is in the form:

$$\frac{dZ^*_{n+1}}{dt_n} = \left(\frac{dZ^*_{n+1}}{dZ^*_{n(j-1)}}\right)^0 \cdot \left(\frac{dZ^*_{n(j-1)}}{dZ^*_{n(j-2)}}\right)^0 \cdots \left(\frac{dZ^*_{n2}}{dZ^*_{n1}}\right)^0 \cdot \left(\frac{dZ^*_{n1}}{dt_n}\right)^0 \tag{12}$$

The recurrent expression (11) furnishes all the initial factors of the product, and this expression is obtained by deriving Eq. (8), in which i = 1:

$$\frac{dZ^*_{n1}}{dt_n} = \phi_1(t^0_{n\,1}) \left[\frac{d\phi_1(-t_n)}{dt_n} \cdot Z_n + \phi_1(-t_n)\cdot\left(\frac{dZ_n}{dt_n}\right)_{t_n^-} - \phi_1(-t_n).B_1.u(t_n) \right] \tag{13}$$

that is, all computations performed and taking the hypothesis of weak signals into account:

$$\left(\frac{dZ^*_{n1}}{dt_n}\right)^0 = \phi_1^0 \left[\left(\frac{dZ}{dt_n}\right)_{t_n^-} - \left(\frac{dZ}{dt_n}\right)_{t_n^+} \right] \tag{14}$$

with

$$\phi_1^0 = \phi_1(t_{n1}^0 - t_n^0)$$

This is in the form:

$$\left(\frac{dZ^*_{n1}}{dt_n}\right)^0 = \phi_1^0 . K'_0 \tag{15}$$

Eq. (12) may thus be written:

$$\frac{dZ^*_{n+1}}{dt_n} = \phi_j^0\left[K_{j-1}+\mathbb{1}\right]\phi^0_{(j-1)}\left[K_{(j-2)}+\mathbb{1}\right]\cdots\ \phi_2^0\left[K_1+\mathbb{1}\right]\phi_1^0\,K'_0 \tag{16}$$

Moving on to the following periods, the general term of the recurrence is in the form:

$$\frac{d\underline{Z}^{*}_{(n+k+1)}}{d\underline{Z}^{*}_{n+k}} = \frac{d\underline{Z}^{*}_{n+k+1}}{d\underline{Z}^{*}_{(n+k)(j-1)}} \cdots\cdots \frac{d\underline{Z}^{*}_{(n+k)1}}{d\underline{Z}^{*}_{(n+k)}} \tag{17}$$

All the factors of this product are independent of the rank of the period under consideration, that is:

$$\frac{d\underline{Z}^{*}_{(n+k+1)}}{d\underline{Z}^{*}_{(n+k)}} = \phi_j^0\left[K_{(j-1)} + \mathbb{1}\right] \cdots\cdots \phi_2^0\left[K_1 + \mathbb{1}\right]\phi_1^0\left[K_0 + \mathbb{1}\right] \tag{18}$$

with $k \geq 1$.

Since the discrete pulse response is being sought, the pulse being applied at instant t_n, one has:

$$\left[\frac{dt_{n+k}}{d\underline{Z}^{*}_{n+k}} = 0\right] \text{ and } K_0 = 0 \tag{19}$$

The discrete state transition matrix is thus:

$$\frac{d\underline{Z}^{*}_{n+k+1}}{d\underline{Z}^{*}_{n+k}} \triangleq \phi(T) = \phi_j^0\left[K_{(j-1)} + \mathbb{1}\right] \cdots\cdots \phi_1^0 \tag{20}$$

Since the disturbance is applied at instant t_n, it should be noted that, by the definition of $\underline{Z}_n^{*}$, one has:

$$\frac{d\underline{Z}^{*}_{n}}{dt_n} \triangleq 0$$

and thus that:

$$\Delta Z_n^* = 0,$$

As a result, therefore:

$$\Delta \underline{Z}^*_{n+m} = \frac{d\underline{Z}^*_{n+m}}{dt_n} \Delta t_n = \phi(mT).K'_0 . \Delta t_n = \phi^m(T).K'_0 . \Delta t_n \tag{21}$$

The equivalent linear system is thus that of Fig. 5.

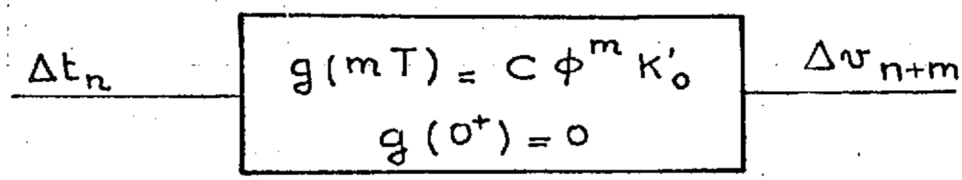


Fig. 5.

Using the diagonal form $\Lambda(T)$ of $\phi(T)$, one has

$$\phi(T) = \mathcal{D}.\Lambda(T).\mathcal{D}^{-1} \quad \text{and} \quad \phi^m(T) = \mathcal{D}.\Lambda^m(T).\mathcal{D}^{-1} \tag{22}$$

as a result of which

$$g(mT) = C.\mathcal{D}.\Lambda^m(T).\mathcal{D}^{-1}.K'_0 \tag{23}$$

C.D. is a line matrix which may be written as:

$$C\mathcal{D} = [c d_1 \quad c d_2 \quad \ldots .]$$

$D^{-1}.K'_0$ is a column matrix which may be written;

$$\mathcal{D}^{-1} \cdot K'_0 = \begin{bmatrix} \alpha k_1 \\ \alpha k_2 \\ \vdots \end{bmatrix}$$

If $\lambda_1, \lambda_2 \ldots$ are the characteristic values (assumed to be distinct) of $\Lambda(t)$, one has:

$$g(mT) = \alpha_1 \cdot \alpha k_1 \cdot \lambda_1^{\ m} + \alpha_2 \cdot \alpha k_2 \cdot \lambda_2^{\ m} + \ldots \qquad \forall m > 0$$

Assuming $\lambda_j = e^{a_j T}$, the continuous pulse response of the equivalent continuous system is such that:

$$g(\underset{t>0}{t}) = \alpha_1 \cdot \alpha k_1 \cdot e^{a_1 t} + \alpha_2 \cdot \alpha k_2 \cdot e^{a_2 t} + \ldots \qquad (24)$$

The z transmittance may thus easily be obtained with the use of ordinary tables, in the form:

$$\frac{\overline{\Delta V}(z)}{\overline{\Delta t'}(z)} = \overline{G}(z, 1) \qquad (25)$$

$\overline{G}(z, 1)$ is used rather than $\overline{G}(z)$ to take into account the fact that $G(0)^+ = 0$; cf. Ref. [5].)

Gain of the Coincidence Modulator

Let V_E equal the control voltage of the coincidence modulator (Fig. 6).

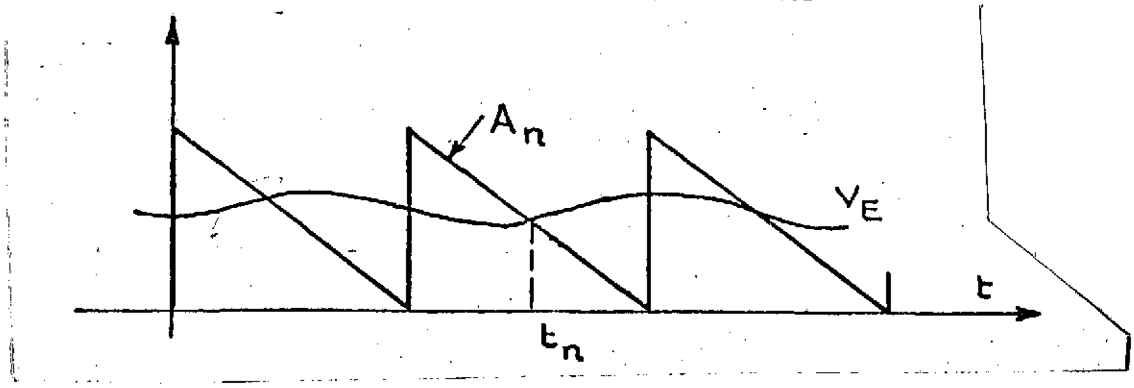


Fig. 6.

Instant t_n is determined by an equation in the form:

$$A_n(t_n) - V_E(t_n) = 0.$$

Ithhas been shown in earlier work [5] that:

$$\Delta t_n = \frac{1}{\left.\frac{dA_n}{dt}\right|_{t_n} - \left.\frac{dV_E}{dt}\right|_{t_n^-}} \Delta V_E(t_n^-) \tag{26}$$

At the limit, one has:

$$\Delta V_E(t_n^-) = \Delta V_E^*(t_n)$$

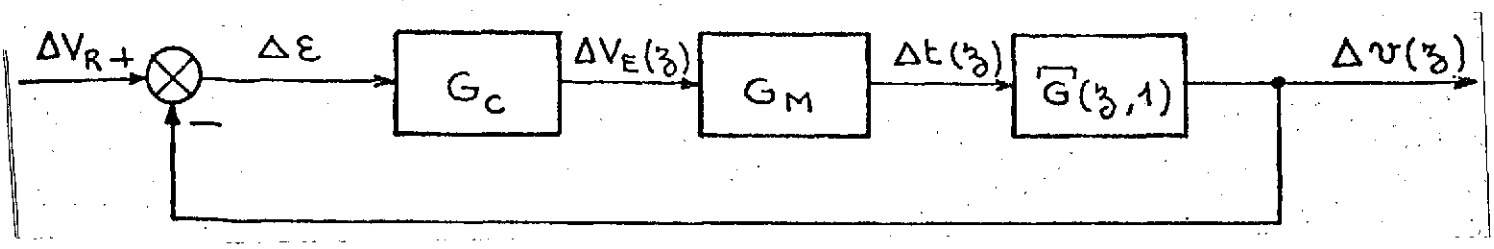
Using the notation

$$G_M = \frac{1}{\left.\frac{dA_n}{dt}\right|_{t_n} - \left.\frac{dV_E}{dt}\right|_{t_n^-}} \tag{27}$$

one has:

$$\Delta t_n = G_M \Delta V_E^*(t_n) \tag{28}$$

and the block diagram of the converter is as follows;



G_C = Const. being the gain of the correcter amplifier.

Fig. 7.

Application to Boost

Let us consider the converter below, for which I and V are chosen as components of the state vector $\underline{Z}$:

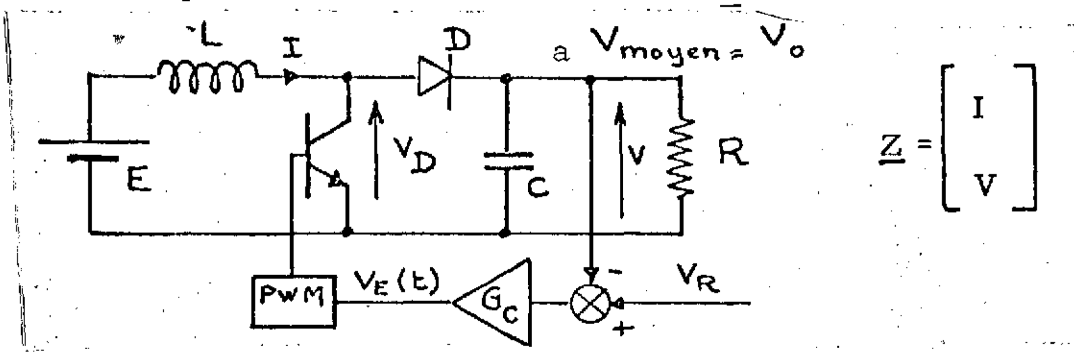


Fig. 8.

Key: a. Mean V = V_o.

The output with which we are concerned is the voltage $V_{(t)}$; thus

$$v_{(t)} = C\,\underline{Z} = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{Z}$$

Let us assume that the width modulation is performed by means of a sawtooth current of amplitude A, that is;

$$A_0(t) = A \frac{t}{T} \qquad (29)$$

by a coincidence modulator, and that the correcter amplifier is a simple amplifier of gain G_C.

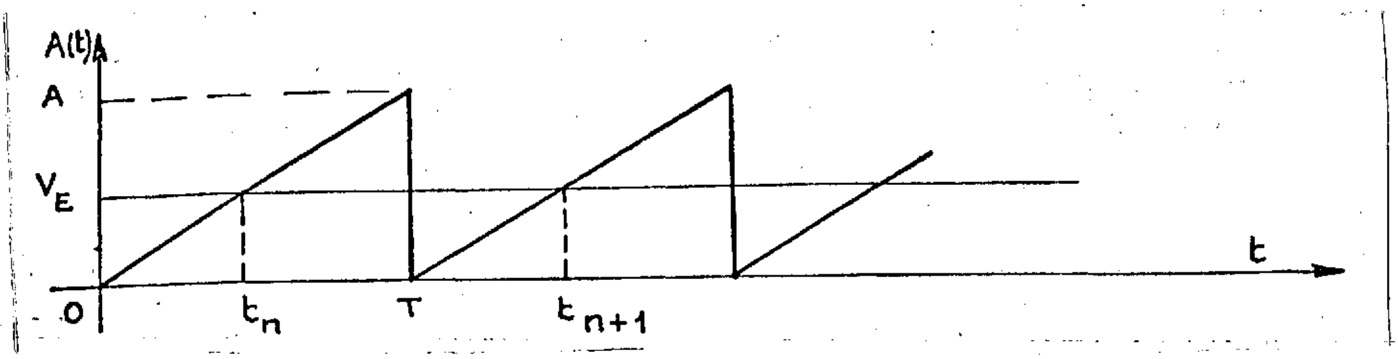
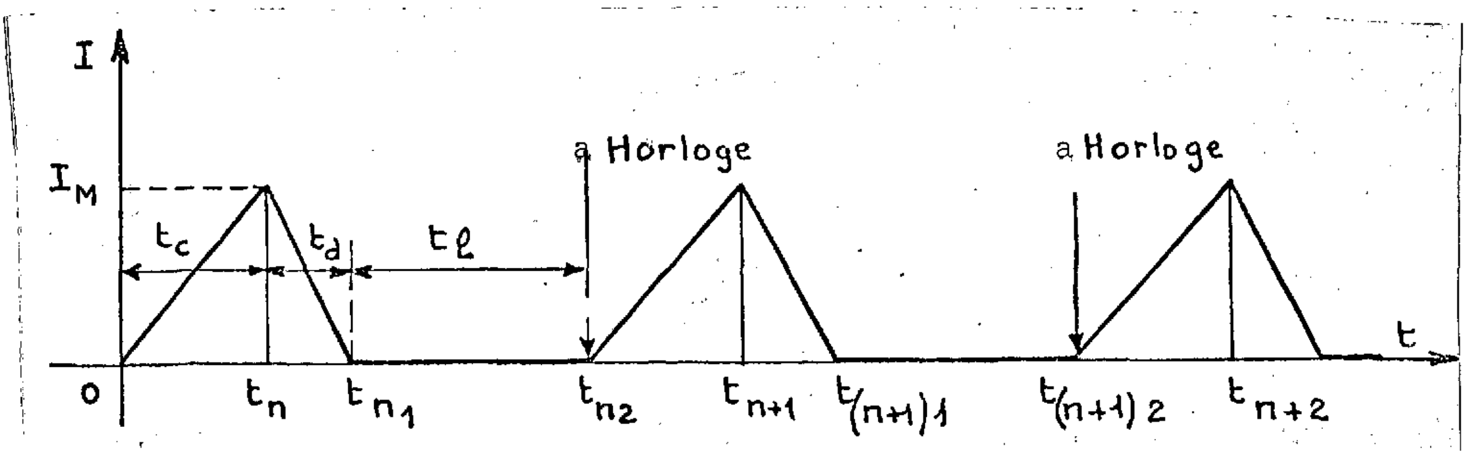


Fig. 9.

Discontinuous Conduction Operation

During one clock period the system undergoes two changes in structure at t_{n1} and t_{n2}.



Key: a. Clock.

It is assumed that the PWM acts at instants t_n, t_{n+1}... and that the clock acts at instants t_{n2}, $t_{(n+1)2}$, ...

The state equations of the system are:

$$\left.\begin{aligned}
&\overset{\circ}{\underline{Z}} = A_1 \underline{Z} + B_1 E = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \underline{Z} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} E \\
&\qquad \text{for } t_n < t < t_{n1} \\
&\overset{\circ}{\underline{Z}} = A_2 \underline{Z} + B_2 E = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \underline{Z} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} E \\
&\qquad \text{for } t_{n1} < t < t_{n2} \\
&\overset{\circ}{\underline{Z}} = A_3 \underline{Z} + B_3 E = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \underline{Z} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} E \\
&\qquad \text{for } t_{n2} < t < t_{n+1} \\
&\left.\begin{aligned} v &= C\,\underline{Z} = [0 \quad 1]\, \underline{Z} \\ i &= C_1 \underline{Z} = [1 \quad 0]\, \underline{Z} \end{aligned}\right\} \forall t
\end{aligned}\right] \tag{30}$$

As a result;

$$\phi_1 = e^{A_1 t} = \begin{bmatrix} \phi_{111} & \phi_{112} \\ \phi_{121} & \phi_{122} \end{bmatrix} =$$

$$\left[\begin{array}{c|c} \frac{1}{\sqrt{LC\omega^2}} e^{-\frac{t}{2RC}} \cos\left(\omega t - \text{arc tg} \frac{1}{2RC\omega}\right) & -\frac{1}{L\omega} e^{-\frac{t}{2\,RC}} \sin \omega\, t \\ \hline \frac{1}{C\omega} e^{-\frac{t}{2RC}} \sin \omega t & \frac{1}{\sqrt{LC\omega^2}} e^{-\frac{t}{2RC}} \cos\left(\omega t + \text{arc tan} \frac{1}{2RC\omega}\right) \end{array}\right] \quad (31)$$

with

$$\omega^2 = \frac{1}{LC} - \frac{1}{4R^2C^2}$$

and

$$\phi_2 = \phi_3 = e^{A_2 t} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{t}{RC}} \end{bmatrix} \quad (32)$$

Given these conditions, it follows that:

$$-\frac{dZ^*_{n1}}{dt_n} = \phi_1 (t_d) \left[\left(\frac{dZ}{dt}\right)_{t_n^-} - \left(\frac{dZ}{dt}\right)_{t_n^+} \right] = \phi_1 (t_d) \left[\left[A_3\right] - \left[A_1\right] \right] Z(t_n)$$

Since $Z_{(t_n)} = \begin{bmatrix} I_M \\ V_0 \end{bmatrix}$, it may be determined from this that;

$$K'_0 = \left[\left[A_3\right] - \left[A_1\right]\right]\begin{bmatrix} I_M \\ V_0 \end{bmatrix} = \begin{bmatrix} \frac{V_0}{L} \\ -\frac{I_M}{C} \end{bmatrix}$$

$$-\frac{dZ^*_{n2}}{dZ^*_{n1}} = \phi_2(t_\ell)\left\{\left[\left(\frac{dZ}{dt}\right)_{(t_{n1}^-)} - \left(\frac{dZ}{dt}\right)_{(t_{n1}^+)}\right]\frac{dt_n}{dZ^*_{n1}} + 1\!\!1\right\}$$

$$= \phi_2(t_\ell)\left\{\left[\left[\left[A_1\right] - \left[A_2\right]\right] Z_{n1} + \left[\left[B_1\right] - \left[B_2\right]\right] E\right] \frac{-C_1}{C_1\left(\frac{dZ}{dt}\right)_{t_{n1}^{0-}}} + 1\!\!1\right\} \qquad (33)$$

$$= \phi_2(t_\ell)\left\{\left[\begin{bmatrix} -\frac{V_0}{L} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}\right] \frac{-\begin{bmatrix} 1 & 0 \end{bmatrix}}{\left(\frac{-V_0}{L} + \frac{E}{L}\right)} + 1\!\!1\right\}$$

$$= \phi_2(t_\ell)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{t_\ell}{RC}} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & e^{-\frac{t_\ell}{RC}} \end{bmatrix}$$

(34)

$$-\frac{dZ^*_{n+1}}{dZ^*_{n2}} = \phi_3(t_c)\left\{\left[\left(\frac{dZ}{dt}\right)_{t_{n2^-}} - \left(\frac{dZ}{dt}\right)_{t_{n2^+}}\right]\frac{dt_{n2}}{dZ^*_{n2}} + 1\!\!1\right\}$$

Since the clock acts at t_{n2}, it follows that $\frac{dt_{n2}}{dZ^*_{n2}} = 0$, and thus;

$$\frac{d\underline{Z}^*_{n3}}{d\underline{Z}^*_{n2}} = \phi_3(t_c) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{t_c}{RC}} \end{bmatrix} \tag{35}$$

One also has:

$$\frac{d\underline{Z}^*_{(n+k)1}}{d\underline{Z}^*_{n+k}} = \phi_1(t_d)$$

from which:

$$\frac{d\underline{Z}^*_{n+k+1}}{d\underline{Z}^*_{n+k}} = \phi_3(t_c)\,\phi_2(t_\ell)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\phi_1(t_d) \triangleq \phi(T) \tag{36}$$

that is:

$$\phi(T) = \begin{bmatrix} 0 & 0 \\ e^{-\frac{t_c+t_\ell}{RC}}\,\phi_{121}(t_d) & e^{-\frac{t_c+t_\ell}{RC}}\,\phi_{122}(t_d) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{37}$$

Since the characteristic values of $\phi(T)$ are 0 and ϕ_{22}, placing the matrices in diagonal form, it follows that:

$$\Delta V_{n+k} = \begin{bmatrix} 0 , & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{\phi_{21}}{\phi_{22}} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \phi_{22}{}^{k} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\phi_{21}}{\phi_{22}} & 1 \end{bmatrix} \begin{bmatrix} \frac{V_0}{L} \\ -\frac{I_M}{C} \end{bmatrix} \Delta V_n$$

$$= \phi_{22}{}^{k} \left[\frac{\phi_{21}}{\phi_{22}} \frac{V_0}{L} - \frac{I_M}{C} \right] \Delta t_n \qquad (38)$$

The pulse response of the equivalent linear system,aãll computations performed, is therefore:

$$\boxed{g\,(t) = \frac{E}{\sqrt{LC}} \left[\frac{V_0}{E} \frac{\sin \omega t_d}{\cos(\omega t_d + \text{arc tan} \frac{1}{2RC\omega})} - \omega t_c \right] e^{-\left[\frac{1}{RC} - \frac{t_d}{T2RC} - \frac{1}{T} \ln \frac{\cos(\omega t_d + \text{arc tan} \frac{1}{2RC\omega})}{\sqrt{LC\omega^2}} \right] t}} \qquad (39)$$

in the form $g(t) = G_B\, e^{-at}$

The discontinuous conduction boost thus behaves as a first order system.

Finally, the pulsed transmittance of the system is:

$$\overline{G}\,(z,\ 1) = G_B \frac{e^{-aT}}{z - e^{-aT}} \qquad (40)$$

resulting in the characteristic equation:

$$z - e^{-aT}(1 - G_C\, G_M\, G_B) = 0 \qquad (41)$$

Note: Taking into account the order of magnitude of the parameters encountered in practice, it may easily be shown that:

$$G_B > 0$$
$$a > 0$$

The z pole for a closed loop is:

$$z_1 = e^{-aT}(1 - G_C\, G_M\, G_B)$$

One has

$$-1 < z_1 < +1 \text{ if one has } (G_C\, G_M\, G_B) > 0:$$

$$\boxed{G_C\, G_M\, G_B < 1 + e^{aT}}$$

The transient behavior is better if $z_1 = 0$, that is, if:

$$G_C\, G_M\, G_B = 1$$

Numerical Example

Let the "boost" regulator of Fig. 8 be such that:

$T = 10^{-5}$ s
$L = 2 \cdot 10^{-5}$ H
$C = 4 \cdot 10^{-4}$ F
$R = 20\ \Omega$
$V_0 = 28$ V
$E = 20$ V

With these values the regulator functions in discontinuous conduction. Thus

$$\omega = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} \# \sqrt{\frac{1}{LC}} = 11,200$$

The system may easily be set up so that $t_c \neq 2.83$ µsec
$t_d \neq 7.07$ µsec
$t_\ell \neq 0.1$ µsec

from which, applying Eq. (39):

$$G_B \neq 17,740 \ .$$

In addition, noting that

$$e^{-aT} = \phi_{22}(T)$$

one has

$$e^{-aT} \neq 0.995$$

One has:

$$\left.\frac{dV}{dt}\right|_{t_n^-} = C \left.\frac{dZ}{dt}\right|_{t_n^-} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} I_M \\ V_0 \end{bmatrix} + \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix} \right] = -\frac{V_0}{RC}$$

from which

$$G_M = \frac{1}{\frac{A}{T} - G_C \left[\left.\frac{dV_R}{dt}\right|_{t_n^-} - \left.\frac{dV}{dt}\right|_{t_n^-} \right]}$$

Since y_R = Const.,

$$G_M = \frac{1}{\frac{A}{T} - G_C \frac{V_0}{RC}}$$

and as a result

$$G_C \, G_M = \frac{1}{\frac{A}{G_C \cdot T} - \frac{V_0}{RC}}$$

The z pole for a closed loop is therefore

$$z_1 = e^{-aT} \left(1 - \frac{G_B}{\frac{A}{G_C T} - \frac{V_0}{RC}} \right)$$

Fig. 10 shows the variations in z_1 as a function of $\frac{G_C}{A}$ for different values of the load corresponding to I_{max}, $I_{max}/2$ and $I_{max}/10$.

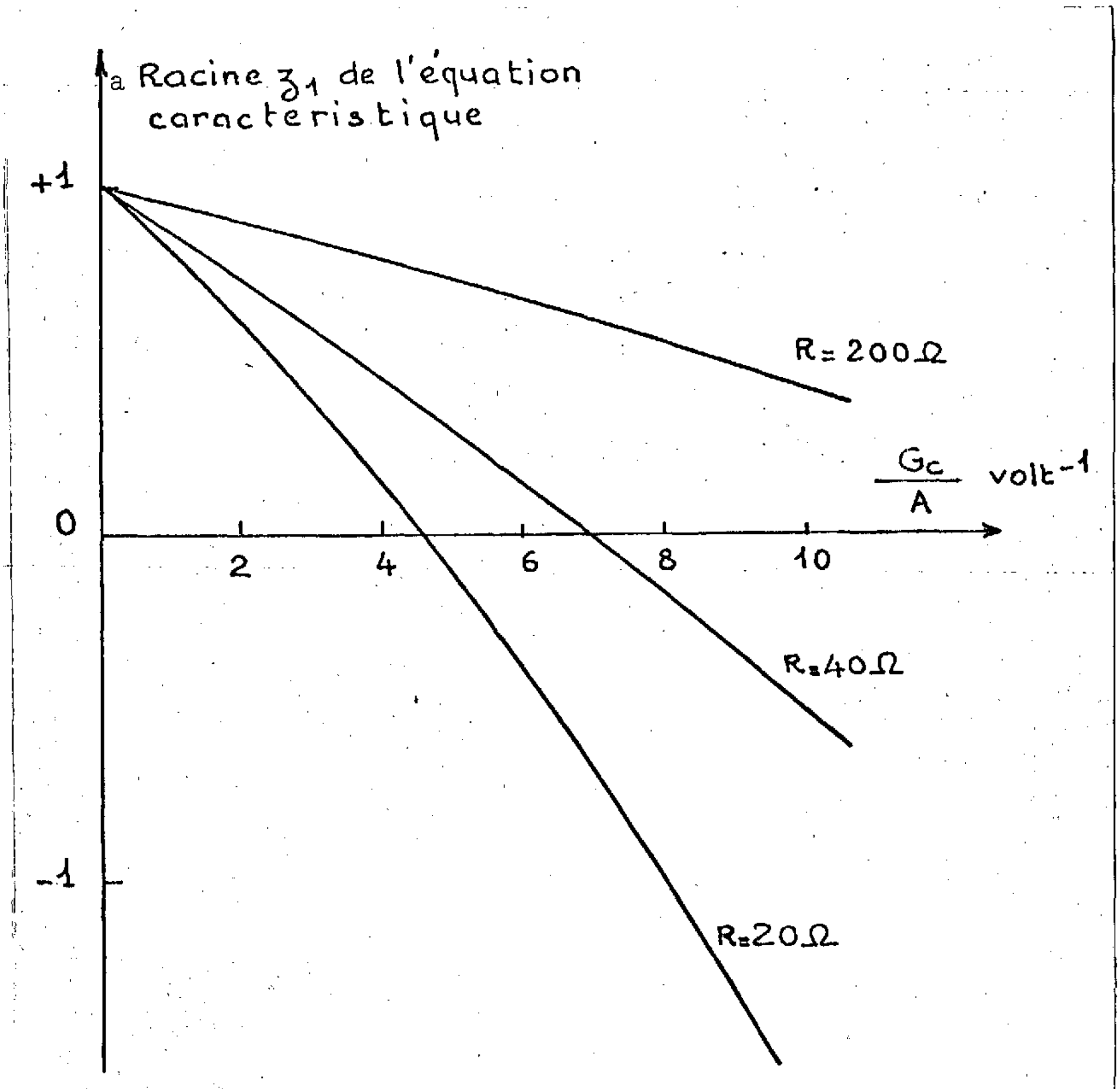


Fig. 10.
Key: a. Root z_1 of characteristic equation.

Conclusion

The method which has just been described makes it possible to put the dynamic behavior of PWM converters in equation form systematically and unproblematically and to determine their pulsed transmittance for any given structure of mode of operation.

The simplifications which may be made for practical use are performed a posteriori and their range is thus quite clear.

The various quantities which may be subsumed by gains and which occur in the equations may be determined either experimentally or by computation.

REFERENCES

1. Wester, G.W., and Middlebrook, R.D., Low frequency characterization of switched DC-DC converters. California Institute of Technology, Pasadena, California 91109.

2. Fallside, F., "Ripple instability in closed loop pulse modulation systems including invertor drives," Proc. IEE, 115(1), 208 (Jan. 1968).

3. Babaa, M.H., A study of voltage step down DC to DC regulated converter with hysteretic bistable trigger. Prepared under Research Grant No. NSG 152-61. Supplement 5, Jan. 1967.

4. Lagasse, J., and Mira, C., "Study of recurrence relationships and their applications by the Laboratoire d'Automatique et de ses Applications Spatiales," Proceedings of the IFAC 5th World Congress, Paris, June 12-17, 1972.

5. Prajoux, R., "Setting up a model for the weak signal behavior of some cutoff regulators," Spacecraft Power Conditioning Electronics (April 1972).

APPENDIX 1. Computation of $dZ_n^*(i+1)/dZ_{ni}^*$

Since:

$$\frac{d\emptyset i}{dt} = A_i e^{A_i t} = e^{A_i t} A_i$$

Eq. (9) may be written:

$$\frac{dZ^*_{n(i+1)}}{dZ^*_{ni}} = \emptyset_{(i+1)}(t^\circ_{n(i+1)} - t_{ni}) - \left[A_{(i+1)} \cdot Z_{ni} \cdot \frac{dt_{ni}}{dZ^*_{ni}} + \frac{dZ_{ni}}{dZ^*_{ni}} - B_{(i+1)} \cdot u(t_{ni}) \cdot \frac{dt_{ni}}{dZ^*_{ni}}\right]$$

that is, over a span of continuous operation:

$$\left(\frac{dZ^*_{n(i+1)}}{dZ^*_{ni}}\right)^\circ = \emptyset^\circ_{(i+1)}\left[- A_{(i+1)} \cdot Z^\circ_{ni}\left(\frac{dt_{ni}}{dZ^*_{ni}}\right)_{t^\circ_{ni}} + \left(\frac{dZ_{ni}}{dZ^*_{ni}}\right)_{t^\circ_{ni}} - B_{(i+1)} \cdot u_{t^\circ ni} \cdot \left(\frac{dt_{ni}}{dZ^*_{ni}}\right)_{t^\circ_{ni}}\right]$$

with

$$\emptyset^\circ_{(i+1)} = \emptyset_{(i+1)}(t^\circ_{n(i+1)} - t^\circ_{ni})$$

Noting that:

$$\left(\frac{dZ}{dt}\right)_{t^\circ_{ni+}} = A_{(i+1)} Z^\circ_{ni} + B_{(i+1)} \cdot u(t^\circ_{ni})$$

one has

$$\left(\frac{dZ^*_{n(i+1)}}{dZ^*_{ni}}\right)^\circ = \emptyset^\circ_{(i+1)}\left[-\left(\frac{dZ}{dt}\right)_{t^\circ_{ni}+} \cdot \left(\frac{dt_{ni}}{dZ^*_{ni}}\right)_{t^\circ_{ni}} + \left(\frac{dZ_{ni}}{dZ^*_{ni}}\right)_{t^\circ_{ni}}\right]$$

Computing $\frac{dt_{ni}}{dZ^*_{ni}}$ and $\frac{dZ_{ni}}{dZ^*_{ni}}$ separately (Appendices 2 and 3), this equation may be written:

$$\left(\frac{d\underline{Z}^*_{n(i+1)}}{d\underline{Z}^*_{ni}}\right)^\circ = \emptyset^\circ_{i+1}\left[\left[\left(\frac{d\underline{Z}}{dt}\right)_{t^{\circ -}_{ni}} - \left(\frac{d\underline{Z}}{dt}\right)_{t^{\circ +}_{ni}}\right]\left(\frac{dt_{ni}}{d\underline{Z}^*_{ni}}\right)_{t^\circ_{ni}} + \mathbb{1}\right]$$

APPENDIX 2. Computation of $dt_{ni}/d\underline{Z}^*_{ni}$

In the vicinity of t_{ni}, at any given instant t one has:

$$f_i(t) = \emptyset_i(t-t^\circ_{ni})\cdot\underline{Z}^*_{ni} + \int_{t^\circ_{ni}}^{t} \emptyset_i(t-\lambda)\cdot B_i\cdot u(\lambda)\, d\lambda$$

In addition, the equation:

$$h_i\ (\underline{Z}_{ni}) = 0$$

defining the switching instant t_{ni} becomes:

$$C_i\cdot Z(t_{ni},\underline{Z}^*_{ni}) = 0$$

that, is, differentiating:

$$C_i\left[\frac{\partial fi}{\partial t_{ni}}\ \frac{dt_{ni}}{d\underline{Z}^*_{ni}} + \frac{\partial fi}{\partial \underline{Z}^*_{ni}}\right] = 0$$

from which one may derive:

$$\frac{dt_{ni}}{d\underline{Z}^*_{ni}} = \frac{-C_i\frac{\partial fi}{\partial \underline{Z}^* ni}}{C_i\ \frac{\partial fi}{\partial tni}}$$

$$= \frac{-C_i\cdot\ \emptyset_i(t_{ni}-t^\circ_{ni})}{C_i\left[A_i\cdot\emptyset_i(t_{ni}-t^\circ_{ni})\cdot\underline{Z}^*_{ni}+\emptyset_i(t_{ni}-t_{ni})\cdot B_i\cdot u(t_{ni})\right]}$$

Knowing that $\emptyset_i(0) = 1$ over a span of continuous operation, this equation amounts to:

$$\left(\frac{dt_{ni}}{d\underline{Z}^*_{ni}}\right)_{t^\circ_{ni}} = \frac{-\ C_i}{C_i\cdot\left(\frac{d\underline{Z}}{dt}\right)_{t^{\circ-}_{ni}}}$$

APPENDIX 3. Computation of $\left| dZ_{ni}/dZ^*_{ni} \right|$

In the vicinity of t_{ni}, at any given instant t one has:

$$f_i(t) = \emptyset_i(t - t^\circ_{ni}) \cdot \underline{Z}^*_{ni} + \int_{t^\circ_{ni}}^{t} \emptyset_i(t-\lambda) \cdot B_i \cdot u(\lambda) \cdot d\lambda$$

from which:

$$\frac{d\underline{Z}_{ni}}{d\underline{Z}^*_{ni}} = \frac{df_i(t)}{d\underline{Z}^*_{ni}} = \frac{d\,\emptyset_i(t_{ni}-t^\circ_{ni})}{d\,t_{ni}}\,\underline{Z}^*_{ni}\,\frac{dt_{ni}}{d\underline{Z}^*_{ni}} + \emptyset_i\,(t_{ni}-t^\circ_{ni})$$

$$+ \emptyset_i(t_{ni}-t_{ni}) \cdot B_i \cdot u(t_{ni}) \cdot \frac{dt_{ni}}{d\underline{Z}^*_{ni}}$$

since:

$$\emptyset_i(0) = \mathbb{1}$$

$$\frac{d\,\emptyset_i(t_{ni}-t^\circ_{ni})}{dt_{ni}} = A_i\,\emptyset(t_{ni}-t^\circ_{ni})$$

Over continuous operation, the result is that:

$$\left(\frac{d\underline{Z}_{ni}}{d\underline{Z}^*_{ni}}\right)^\circ = \left[A_i \cdot \underline{Z}^*_{ni} + B_i \cdot u(t_{ni})\right] \frac{dt_{ni}}{d\underline{Z}^*_{ni}} + \mathbb{1}$$

that is:

$$\left(\frac{d\underline{Z}_{ni}}{d\underline{Z}^*_{ni}}\right)^\circ = \left(\frac{dZ}{dt}\right)_{t^{\circ -}_{ni}} \cdot \frac{dt_{ni}}{d\underline{Z}^*_{ni}} + \mathbb{1}$$